

KUNS-2072
arXiv:0706.4471
June 2007

Notes on D-branes and dualities in (p, q) minimal superstring theory

Hiroataka Irie*

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

abstract

We study boundary states in (p, q) minimal superstring theory, combining the explicit form of matter wave functions. Within the modular bootstrap framework, Cardy states of (p, q) minimal superconformal field theory are completely determined in both cases of the different supercharge combinations, and the remaining consistency checks in the super-Liouville case are also performed. Using these boundary states, we determine the explicit form of FZZT- and ZZ-brane boundary states both in type 0A and 0B GSO projections. Annulus amplitudes of FZZT branes are evaluated and principal FZZT branes are identified. In particular, we found that these principal FZZT branes do not satisfy Cardy's consistency conditions for each other and play a role of order/disorder parameters of the Kramers-Wannier duality in spacetime of this superstring theory.

*E-mail: irie@gauge.scphys.kyoto-u.ac.jp

1 Introduction and summary

Noncritical (super)string theories have been investigated as a useful laboratory of critical string theory and have been discussed in various contexts: the worldsheet description [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], matrix models [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and the string field theoretical descriptions [33, 34, 35, 36, 37, 38, 39]. They have fewer degrees of freedom and we can make a detail study of many important properties shared with their critical counterparts. Moreover, since these string theories are described with many different formulations, we can investigate various aspects of stringy phenomena.

Here we further study (p, q) minimal superstring theory from the conformal field theory (or Liouville theory) approach. Minimal superstring theory is one of the most tractable superstring theories among these noncritical superstring theories. Its worldsheet description is defined with $\mathcal{N} = 1$ super Liouville field theory [1, 2, 3] coupled to $\mathcal{N} = 1$ (p, q) minimal superconformal field theory [40, 41, 42, 43] with type 0 GSO projection (see [29, 15] for its basic properties). From the recent developments in Liouville field theory without boundaries [5, 6, 7, 8] and with boundaries [9, 10, 11], the boundary states of super Liouville field theory were obtained in [12, 13], and in the framework of minimal superstring theory they have been extensively studied in [15, 32, 17], where the disk amplitudes of corresponding D-branes were explicitly evaluated [15] and the pure-supergravity case, $(p, q) = (2, 4)$, was studied including annulus amplitudes [32, 17]. For further investigation of these D-branes, however, we need to know all the Cardy states of (p, q) minimal superconformal field theory with different supercharge combinations of left and right:

$$(G_r - i\eta \bar{G}_{-r})|B; \eta\rangle = 0 \quad (\eta = \pm 1), \quad (1.1)$$

and we need to combine Cardy states of each SCFT in practice.

As is known from the original works on the conformally invariant boundary states [44, 45, 46], Cardy states have a one-to-one correspondence with the highest weight representations of the open-channel Virasoro algebra. As is briefly reviewed in section 2.1, Cardy states have the following form in this superconformal case:¹

$$|h_{NS\pm}\rangle = \sum_{i \in NS} B_{h_{NS}}^{i_{NS}} |i_{NSNS}; \eta = -1\rangle\rangle \pm \sum_{i \in R} B_{h_{\bar{NS}}}^{i_R} |i_{RR}; \eta = -1\rangle\rangle,$$

¹ Since open strings between opposite η boundaries are in Ramond sector, we make use of the convention that Cardy states of $\eta = -1$ ($+1$) are labeled with the highest weight in the NS (R) sector of the open-string super-Virasoro algebra.

$$|h_{R\pm}\rangle = \sum_{i \in NS} B_{h_R}{}^{i\widetilde{NS}} |i_{NSNS}; \eta = +1\rangle\rangle \pm \sum_{i \in \widetilde{R}} B_{h_{\widetilde{R}}}{}^{i\widetilde{R}} |i_{RR}; \eta = +1\rangle\rangle. \quad (1.2)$$

We often use a notation like j_{RR} in this paper to indicate the sector to which an index j belongs, and $|j; \eta\rangle\rangle$ is a corresponding Ishibashi state [45]. Three kinds of these wave functions B_h^i (h and i are among NS, \widetilde{NS} and R sectors) are known from the work of [47],² and they are written with the modular matrices S_h^i of this theory [48] as

$$B_{h_{NS}}{}^{i_{NS}} = \frac{1}{\sqrt{2}} \frac{S_{h_{NS}}{}^{i_{NS}}}{\sqrt{S_{0_{NS}}{}^{i_{NS}}}}, \quad B_{h_{\widetilde{NS}}}{}^{i_R} = \frac{1}{2^{3/4}} \frac{S_{h_{\widetilde{NS}}}{}^{i_R}}{\sqrt{S_{0_{\widetilde{NS}}}{}^{i_R}}}, \quad B_{i_R}{}^{h_{\widetilde{NS}}} = \frac{S_{i_R}{}^{h_{\widetilde{NS}}}}{\sqrt{S_{0_{NS}}{}^{i_{NS}}}}. \quad (1.3)$$

On the other hand, the form of the remaining wave functions $B_{h_{\widetilde{R}}}{}^{i_{\widetilde{R}}}$ is still not known. One of the reasons why these wave functions $B_{h_{\widetilde{R}}}{}^{i_{\widetilde{R}}}$ have not been obtained is that they do not satisfy such a simple relation with modular matrices S_h^i like (1.3). Since any corresponding characters (those in \widetilde{R} sector) always vanish due to the fermion zero-modes in the cylindrical geometry (except for the Ramond ground states), there is not such a modular matrix S_h^i among these \widetilde{R} characters.

If one considers (p, q) minimal superconformal field theory as only a single SCFT, this does not cause any problems in studying the boundary states as in the literature [49]. It is because we impose the spin-model GSO projection [41] ($\Gamma = -1$ on RR states) and the above remaining states are always projected out. On the other hand, the case of the superstring theory is not so. Since superstring theory is a combined superconformal field theory (Liouville, matters and ghosts), the type 0 GSO projection cannot eliminate these contributions [29, 15]. So this is the first task of this paper: We will completely determine all the wave functions of Cardy states, including the remaining wave functions, in the case of (p, q) minimal superconformal field theory (section 2.2).

Actually, the way to obtain these remaining wave functions is very simple. All we have to do is to consider the OPE algebra with the simplest degenerate primaries $(1, 2)_+$ [40] in the sense of open (or chiral) superconformal field theory and to consider the Cardy equations obtained from the inner products with the Ramond Cardy state $|(1, 2)_{R+}\rangle$. The plus symbol in the subscript of $(1, 2)_+$ means that this is a primary operator of positive chirality. Essentially we use the following fusion rule that could be read off from the super Coulomb gas formalism in Ramond sector [43]:

$$\mathcal{N}_{(1,2)_+, (k,l)_+}{}^{(r,s)} = \delta_{(k,l+1)_+}{}^{(r,s)} + \delta_{(k,l-1)_-}{}^{(r,s)}, \quad (1.4)$$

² The Cardy states of Ramond ground states (denoted as $|\theta_{R\pm}\rangle$ in this paper) are not considered in [47]. This means that only odd models are considered. In this paper, we also derive the formula of this case.

where operators of both positive and negative chiralities come into the relations. This kind of fusion rule is also found in the super-Liouville cases [12] and we actually show that all of the results obtained within the conformal bootstrap methods [12] are consistent with our procedure. This means that all the information about boundary states can be extracted from the modular bootstrap methods also in the case of $\mathcal{N} = 1$ superconformal field theory.³

In section 2.3, we present the complete form of the boundary states in both cases of 0B and 0A theory, and investigate their basic properties. As in the case of super Liouville field theory [12], the wave functions of $\eta = -1/+1$ Cardy states in (p, q) minimal superconformal field theory are not symmetric under the transformation of $(-1)^{f_R}$ (f_R is the worldsheet fermion number), even though the superconformally invariant boundary conditions (1.1) of $\eta = \pm 1$ are exchanged for each other. The superstring Cardy state is also not an exception but we show that one of the branes obtained from the transformation of $(-1)^{f_R}$ comes to play a role of fundamental degrees of freedom, principal FZZT branes [15]. The principal $\eta = -1/+1$ FZZT branes of this theory are simply related under the transformation of $(-1)^{f_R}$, and actually we show that they do not satisfy the Cardy equations among each other. This is reminiscent of order/disorder parameters of the Kramers-Wannier duality. We actually argue that they are not mutually local in the superstring spacetime, by evaluating annulus amplitudes (section 3).

The organization of this paper is as follows: In section 2, we summarize the Cardy states of this theory. After we briefly review the definition of Cardy states in section 2.1, we show the way to derive the remaining wave functions in section 2.2 and the boundary states of our superstring theory are discussed in section 2.3. In section 3, we evaluate various annulus amplitudes of FZZT branes. Section 4 is devoted to conclusion and discussion.

2 Boundary states of (p, q) minimal superstring theory

2.1 Cardy states in superconformal field theory

We now recall the definition of Cardy states [44] in the SCFT case to fix our notation. This case (but only odd models) was studied under the spin-model GSO projection [47]. The following discussion does not require any restrictions, and any GSO projections are not

³Of course, we need to perform the conformal bootstrap method to know some relations with variables in the boundary action (e.g., boundary cosmological constants ζ in Liouville theory).

imposed from the beginning.

Superconformally invariant boundary states $|B; \eta\rangle$ are defined as

$$(L_n - \bar{L}_{-n})|B; \eta\rangle = (G_r - i\eta \bar{G}_{-r})|B; \eta\rangle = 0, \quad (\eta = \pm 1; n \in \mathbb{Z}, r \in \mathbb{Z} + \nu). \quad (2.1)$$

Here L_n (\bar{L}_n) and G_r (\bar{G}_r) are the left (right) handed super-Virasoro generators that satisfy

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n+m,0} \\ [L_n, G_r] &= (n - r)G_{n+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}, \end{aligned} \quad (2.2)$$

and $\nu = 1/2$ (or $\nu = 0$) when $|B; \eta\rangle$ is in NSNS (or RR) sector. The closed string Hilbert space can be expanded into irreducible Verma modules of super-Virasoro algebra or super-conformal families

$$\mathcal{H}^{(c)} = \bigoplus_{i \in NSNS, RR} \mathcal{V}_i \otimes \bar{\mathcal{V}}_i \equiv \bigoplus_{i \in NSNS, RR} [\phi_i(z, \bar{z})]. \quad (2.3)$$

When the left and right Verma modules are isomorphic, $\mathcal{V}_i \cong \bar{\mathcal{V}}_i$, one can find the Ishibashi state [45] in these irreducible Verma modules,

$$|i; \eta\rangle\rangle = \sum_N |i; N\rangle \otimes U_\eta A \overline{|i; N^*\rangle}, \quad (2.4)$$

which satisfies (2.1). Here A is an anti-unitary operator that commutes with the above super-Virasoro generators, U_η is an automorphism of irreducible Verma module $\bar{\mathcal{V}}_i$,

$$U_\eta \bar{L}_n U_\eta^{-1} = \bar{L}_n, \quad U_\eta \bar{G}_r U_\eta^{-1} = i\eta \bar{G}_r (-1)^{f_R}, \quad U_\eta^\dagger = U_\eta^{-1}, \quad (2.5)$$

and $\overline{|i; N^*\rangle}$ is a hermitian conjugation (defined by $\bar{G}_r^\dagger = \bar{G}_{-r}$, $\bar{L}_n^\dagger = \bar{L}_{-n}$) of a dual base $\overline{\langle i; N^*|}$,⁴

$$\overline{\langle i; N^*| j; M \rangle} = \overline{\langle i| j \rangle} \delta_{N,M}. \quad (2.6)$$

Note that the chirality Γ on each states is given as⁵

$$\Gamma |i_{NSNS}; \eta\rangle\rangle = + |i_{NSNS}; \eta\rangle\rangle, \quad \Gamma |i_{RR}; \eta\rangle\rangle = \eta |i_{RR}; \eta\rangle\rangle, \quad \Gamma |\theta_{RR}; \eta\rangle\rangle = + |i_{RR}; \eta\rangle\rangle, \quad (2.7)$$

⁴In the case of unitary CFTs, we can take orthonormal bases; Since we have to treat nonunitary CFTs in general, we use dual bases. Our construction of these Ishibashi states is based on Watt's technique noted in [50].

⁵We can also define the chirality of the Ramond-Ramond ground state as $\Gamma |\theta_{RR}; \eta\rangle\rangle = - |i_{RR}; \eta\rangle\rangle$.

and states of different η are related as $|i; -\eta\rangle\rangle = (-1)^{f_R} |i; \eta\rangle\rangle$. Also note that RR zero-mode representations in these Ishibashi states are given as

$$|j_{RR}; \eta\rangle\rangle = |j+\rangle \otimes \overline{|j+\rangle} + i\eta |j-\rangle \otimes \overline{|j-\rangle} + \dots, \quad (2.8)$$

and $\langle j+|j+\rangle = \langle j-|j-\rangle = 1$.

Annulus amplitudes between these Ishibashi states are then given with the moduli parameter $q = \bar{q} = e^{-2\pi t}$ as

$$\langle\langle i; \eta | q^{\frac{1}{2}(L_0 - \frac{c}{16})} \bar{q}^{\frac{1}{2}(\bar{L}_0 - \frac{c}{16})} | j; \eta' \rangle\rangle = \mathcal{G}_{ij} \text{tr}_i(\eta\eta')^f q^{L_0 - \frac{c}{16}}. \quad (2.9)$$

Here we denote $\mathcal{G}_{ij} \equiv \langle\phi_i | \phi_j\rangle$ as the Zamolodchikov metric of superconformal primary fields $\phi_i(z, \bar{z})$. It is convenient to introduce dual superconformal primary fields $\phi_{i^*} \equiv \mathcal{G}^{ij} \phi_j$, then we can write the formal completeness relation as

$$\mathbf{1}_\eta = \sum_{i,j} |i, \eta\rangle\rangle \mathcal{G}^{ij} \langle\langle j, \eta| = \sum_i |i, \eta\rangle\rangle \langle\langle i^*, \eta|. \quad (2.10)$$

Considering this relation, closed-channel amplitudes between general boundary states $|\alpha; \eta\rangle \equiv \sum_i |i; \eta\rangle\rangle \langle\langle i^*; \eta| \alpha\rangle$ are expressed as

$$\langle\alpha; \eta | q^{\frac{1}{2}(L_0 - \frac{c}{16})} \bar{q}^{\frac{1}{2}(\bar{L}_0 - \frac{c}{16})} | \beta; \eta' \rangle = \sum_i \langle\alpha | i; \eta\rangle\rangle \langle\langle i^*; \eta' | \beta\rangle \text{tr}_i(\eta\eta')^f q^{L_0 - \frac{c}{16}}. \quad (2.11)$$

On the other hand, open channel amplitudes are given as a sum over the open channel Hilbert space $\mathcal{H}_{\alpha\beta}^{(o)}$ with the boundaries. Therefore they must be expanded into a sum of Virasoro characters with non-negative integer $n_{\alpha\beta}^h$ [44]:

$$\text{tr}_{\mathcal{H}_{\alpha\beta}^{(o)}} \tilde{q}^{L_0 - \frac{c}{16}} = \sum_h n_{\alpha\beta}^h \text{tr}_h \tilde{q}^{L_0 - \frac{c}{16}}, \quad (2.12)$$

with $\tilde{q} = e^{-2\pi/t}$. Note that the label h runs among irreducible Virasoro primary states belonging to $NS\pm$ or $R\pm$ sector in open channel [47]. Comparing both expressions (2.11) and (2.12), we obtain

$$\begin{aligned} 0 &= \sum_{i \in NSNS, RR} \left(\langle\alpha | i; \eta\rangle\rangle \langle\langle i^*; \eta | \beta\rangle - \sum_{h \in NS\pm} n_{\alpha\beta}^h S_h^i \right) \text{tr}_i q^{L_0 - \frac{c}{16}}, \\ 0 &= \sum_{i \in NSNS, RR} \left(\langle\alpha | i; \eta\rangle\rangle \langle\langle i^*; -\eta | \gamma\rangle - \sum_{h \in R\pm} n_{\alpha\gamma}^h S_h^i \right) \text{tr}_i (-1)^f q^{L_0 - \frac{c}{16}}, \end{aligned} \quad (2.13)$$

where $\{S_l^i\}$ are modular matrices of the characters under S transformation, $\tau \rightarrow -1/\tau$, which we define as,

$$\begin{aligned}\chi_{h_{NS\pm}}(-1/\tau) &\equiv \sum_i S_{h_{NS\pm}}^i \chi_i(\tau) = \frac{1}{2} \sum_{i_{NS}} S_{h_{NS}}^{i_{NS}} \chi_{i_{NS}}(\tau) \pm \frac{1}{2\sqrt{2}} \sum_{i_R} S_{h_{\overline{NS}}}^{i_R} \chi_{i_R}(\tau), \\ \chi_{h_{R\pm}}(-1/\tau) &\equiv \sum_i S_{h_{R\pm}}^i \chi_i(\tau) = \frac{\sqrt{2}}{2} \sum_{i_{\overline{NS}}} S_{h_R}^{i_{\overline{NS}}} \chi_{i_{\overline{NS}}}(\tau), \\ \chi_{\theta_{R\pm}}(-1/\tau) &\equiv \sum_i S_{\theta_{R\pm}}^i \chi_i(\tau) = \frac{1}{2\sqrt{2}} \sum_{i_{\overline{NS}}} S_{h_R}^{i_{\overline{NS}}} \chi_{i_{\overline{NS}}}(\tau) \pm \frac{1}{2} \delta_{h_R, \theta_R}.\end{aligned}\quad (2.14)$$

Noting that the characters in \tilde{R} sector must vanish due to fermion zero modes, except for the Ramond ground states (here we define $\chi_{\theta_R}^{\tilde{R}}(\tau) = 1$), we obtain the following Cardy equations:

$$\begin{aligned}0 &= \langle \alpha | i; \eta \rangle \langle \langle i^*; \eta | \beta \rangle - \sum_{h \in NS\pm} n_{\alpha\beta}^h S_h^i, \quad (i \in NSNS, RR), \\ 0 &= \langle \alpha | i; \eta \rangle \langle \langle i^*; -\eta | \gamma \rangle - \sum_{h \in R\pm} n_{\alpha\gamma}^h S_h^i, \quad (i \in NSNS), \\ 0 &= \langle \alpha | \theta_{RR}; \eta \rangle \langle \langle \theta_{RR}^*; -\eta | \gamma \rangle - \sum_{h=\theta_{R\pm}} n_{\alpha\gamma}^{\theta_R}.\end{aligned}\quad (2.15)$$

Then, following the usual procedure (identifying $n_{\alpha\beta}^h$ with the fusion number $\mathcal{N}_{\alpha\beta}^h$ [44] under the Verlinde formula [51] as a non-negative integer valued matrix representation of fusion algebra [50] and considering the trivial relation $\mathcal{N}_{0_{NS+\beta}}^h = \delta_\beta^h$), we can define the Cardy states $|h_{NS\pm}\rangle$ and $|h_{R\pm}\rangle$ for each Virasoro highest weight $h_{NS\pm}$ and $h_{R\pm}$ in open channel:

$$\begin{aligned}|h_{NS\pm}\rangle &= \frac{1}{\sqrt{2}} \sum_{i \in NS} \frac{S_{h_{NS}}^{i_{NS}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = -1\rangle \pm \frac{1}{2^{3/4}} \sum_{i \in R} \frac{S_{h_{\overline{NS}}}^{i_R}}{\sqrt{S_{0_{\overline{NS}}}^{i_R}}} |i_{RR}; \eta = -1\rangle, \\ |h_{R\pm}\rangle &= \sum_{i \in NS} \frac{S_{h_R}^{i_{\overline{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle \pm \frac{\sqrt{2}}{2^{3/4}} \sum_{i \in R} \frac{\psi_{h_{\tilde{R}}}^{i_{\tilde{R}}}}{\sqrt{S_{0_{\overline{NS}}}^{i_R}}} |i_{RR}; \eta = +1\rangle, \\ |\theta_{R\pm}\rangle &= \frac{1}{2} \sum_{i \in NS} \frac{S_{\theta_R}^{i_{\overline{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle \pm \frac{1}{2^{1/4}} \sum_{i \in R} \frac{\psi_{h_{\tilde{R}}}^{i_{\tilde{R}}}}{\sqrt{S_{0_{\overline{NS}}}^{i_R}}} |i_{RR}; \eta = +1\rangle,\end{aligned}\quad (2.16)$$

with $\psi_{\theta_{\tilde{R}}}^{\theta_{\tilde{R}}} = 1$ ($= S_{\theta_{\tilde{R}}}^{\theta_{\tilde{R}}}$) for the Cardy state of the Ramond ground state, $|\theta_{R\pm}\rangle$. Here we denote the identity operator as “0”, and $\sqrt{S_{0_{NS}}^{i_{NS}}}$ is formally defined as it satisfies

$$\sqrt{S_{0_{NS}}^{i_{NS}}} \cdot \sqrt{S_{0_{NS}}^{i_{NS}^*}} \equiv S_{0_{NS}}^{i_{NS}}, \quad \sqrt{S_{0_{\overline{NS}}}^{i_R}} \cdot \sqrt{S_{0_{\overline{NS}}}^{i_R^*}} \equiv S_{0_{\overline{NS}}}^{i_R}.\quad (2.17)$$

The normalization of ψ is chosen as above for later convenience.

Here we consider the spin-model GSO projection ($\Gamma = -1$ on RR sector) [41] for odd models. In this case, the last wave functions ψ_h^i of $|h_\pm\rangle$ can be consistently dropped since all the characters in \tilde{R} sector vanish (see also the Verlinde formula of this case [52]). This gives the previous result [47]:⁶

$$\begin{aligned} |h_{NS\pm}\rangle &= \frac{1}{\sqrt{2}} \sum_{i \in NS} \frac{S_{h_{NS}}^{i_{NS}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = -1\rangle \pm \frac{1}{2^{3/4}} \sum_{i \in R} \frac{S_{h_{\overline{NS}}}^{i_R}}{\sqrt{S_{0_{\overline{NS}}}^{i_R}}} |i_{RR}; \eta = -1\rangle \\ |h_{R+}\rangle &= \sum_{i \in NS} \frac{S_{h_R}^{i_{\overline{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle. \end{aligned} \quad (2.18)$$

For even models, since we should be careful about the Ramond ground state $|\theta_{R\pm}\rangle$ and need to know the remaining wave functions ψ_h^i , we will discuss this case in next subsection.

2.2 The wave functions ψ in each SCFT

In this subsection, we determine the remaining wave functions ψ in each superconformal field theories. As we noted in section 1, we consider the following fusion rule:

$$n_{(1,2)_+, (k,l)_+}^{(r,s)} = 2\mathcal{N}_{(1,2)_+, (k,l)_+}^{(r,s)} = 2\delta_{(k,l+1)_+}^{(r,s)} + 2\delta_{(k,l-1)_-}^{(r,s)}. \quad (2.19)$$

of the simplest degenerate primary $(1,2)_+$ of Virasoro algebra [40]. The factor 2 in front of $\mathcal{N}_{i,j}^k$ comes from the degeneracy of Ramond zero-mode representations.⁷ We first reconsider the case of super Liouville field theory and see that this procedure actually reproduces the results of [12]. We then discuss the case of (p,q) minimal superconformal field theory. We also summarize the corresponding modular matrices in each SCFT.

2.2.1 the super-Liouville case

The corresponding Cardy states were found in [12, 13] and further discussed in [28, 29, 15, 32].⁸ They are expanded in the off-shell Hilbert space [4],

$$\mathcal{H}^{(c)} = \bigotimes_{\nu_{NSNS,RR} > 0} \mathcal{V}_\nu \otimes \widetilde{\mathcal{V}}_\nu = \bigotimes_{\nu_{NSNS,RR} > 0} [e^{(Q/2 + i\nu)\phi(z, \bar{z})}], \quad (2.20)$$

⁶Note that our normalization of RR Ishibashi states is differed by $\sqrt{2}$ from [47], that is $|i_{RR}; \eta = \pm 1\rangle\rangle_{here} = \sqrt{2}|i_{RR}; \eta = \pm 1\rangle\rangle_{there}$

⁷See e.g., [52]. This naturally comes from the Verlinde formula of super-Virasoro algebra.

⁸Note that the case of $\mathcal{N} = 2$ super-Liouville theory was investigated in [53]

and the Cardy states for non-degenerate representations, $\Delta_\sigma = Q^2/8 + \sigma^2/2$, are

$$\begin{aligned}
|\sigma_{NS\pm}\rangle &= \int_0^\infty d\nu \left(\frac{1}{\sqrt{2}} (2 \cosh(2\pi i \sigma \nu)) A_{NS}^{(L)}(\nu) |\nu_{NSNS}; \eta = -1\rangle \rangle \pm \right. \\
&\quad \left. \pm \frac{1}{2^{3/4}} (2 \cosh(2\pi i \sigma \nu)) A_R^{(L)}(\nu) |\nu_{RR}; \eta = -1\rangle \rangle \right), \\
|\sigma_{R\pm}\rangle &= \int_0^\infty d\nu \left((2 \cosh(2\pi i \sigma \nu)) A_{NS}^{(L)}(\nu) |\nu_{NSNS}; \eta = +1\rangle \rangle \pm \right. \\
&\quad \left. \pm \frac{\sqrt{2}}{2^{3/4}} (2i \sinh(2\pi i \sigma \nu)) A_R^{(L)}(\nu) |\nu_{RR}; \eta = +1\rangle \rangle \right), \tag{2.21}
\end{aligned}$$

and those for degenerate representations $\sigma = i(nb + m/b)$ are

$$\begin{aligned}
|(n, m)_{NS\pm}\rangle &= \int_0^\infty d\nu \left(\frac{1}{\sqrt{2}} \left(4 \sinh(\pi n b \nu) \sinh\left(\frac{\pi m \nu}{b}\right) \right) A_{NS}^{(L)}(\nu) |\nu_{NSNS}; \eta = -1\rangle \rangle \pm \right. \\
&\quad \left. \pm \frac{1}{2^{3/4}} \left(4 \sinh\left(\pi n b \nu + \frac{i\pi n}{2}\right) \sinh\left(\frac{\pi m \nu}{b} - \frac{i\pi n}{2}\right) \right) A_R^{(L)}(\nu) |\nu_{RR}; \eta = -1\rangle \rangle \right), \\
|(n, m)_{R\pm}\rangle &= \int_0^\infty d\nu \left(\left(4 \sinh(\pi n b \nu) \sinh\left(\frac{\pi m \nu}{b}\right) \right) A_{NS}^{(L)}(\nu) |\nu_{NSNS}; \eta = +1\rangle \rangle \pm \right. \\
&\quad \left. \pm \frac{\sqrt{2}}{2^{3/4}} \left(4i \sinh\left(\pi n b \nu + \frac{i\pi n}{2}\right) \cosh\left(\frac{\pi m \nu}{b} - \frac{i\pi n}{2}\right) \right) A_R^{(L)}(\nu) |\nu_{RR}; \eta = +1\rangle \rangle \right), \tag{2.22}
\end{aligned}$$

where $A_{NS}^{(L)}(\nu)$ and $A_R^{(L)}(\nu)$ are defined as

$$\begin{aligned}
A_{NS}^{(L)}(\nu) &\equiv \frac{\Gamma(1 - i\nu b) \Gamma(1 - i\nu/b)}{2\pi\nu} \mu^{-i\nu/b} = 1/\sqrt{S_{0NS}^{\nu_{NS}}}, \\
A_R^{(L)}(\nu) &\equiv \frac{\Gamma(1/2 - i\nu b) \Gamma(1/2 - i\nu/b)}{2\pi} \mu^{-i\nu/b} = 1/\sqrt{S_{0\widetilde{NS}}^{\nu_R}}. \tag{2.23}
\end{aligned}$$

The modular bootstrap method, except for the \widetilde{R} wave function ψ , was studied in [12, 13].

The corresponding modular matrices are actually obtained as

$$\begin{aligned}
S_{\sigma_{NS}}^{(L) \nu_{NS}} &= S_{\sigma_{\widetilde{NS}}}^{(L) \nu_R} = S_{\sigma_R}^{(L) \nu_{\widetilde{NS}}} = 2 \cosh(2\pi i \sigma \nu) \\
S_{(n,m)_{NS}}^{(L) \nu_{NS}} &= S_{(n,m)_R}^{(L) \nu_{\widetilde{NS}}} = 4 \sinh(\pi n b \nu) \sinh\left(\frac{\pi m \nu}{b}\right) \\
S_{(n,m)_{\widetilde{NS}}}^{(L) \nu_R} &= 4 \sinh\left(\pi n b \nu + \frac{i\pi n}{2}\right) \sinh\left(\frac{\pi m \nu}{b} - \frac{i\pi n}{2}\right). \tag{2.24}
\end{aligned}$$

form the characters:

$$\chi_\sigma^{(NS)}(\tau) = q^{\frac{\sigma^2}{2}} \chi_0^{(NS)}(\tau), \quad \chi_\sigma^{(\widetilde{NS})}(\tau) = q^{\frac{\sigma^2}{2}} \chi_0^{(\widetilde{NS})}(\tau), \quad \chi_\sigma^{(R)}(\tau) = q^{\frac{\sigma^2}{2}} \chi_0^{(R)}(\tau) \tag{2.25}$$

for non-degenerate representations, and

$$\begin{aligned}
\chi_{(n,m)}^{(NS)}(\tau) &\equiv \chi_{\frac{i}{2}(nb+m/b)}^{(NS)}(\tau) - \chi_{\frac{i}{2}(nb-m/b)}^{(NS)}(\tau), \\
\chi_{(n,m)}^{(\widetilde{NS})}(\tau) &\equiv \chi_{\frac{i}{2}(nb+m/b)}^{(\widetilde{NS})}(\tau) - (-1)^{mn} \chi_{\frac{i}{2}(nb-m/b)}^{(\widetilde{NS})}(\tau), \\
\chi_{(n,m)}^{(R)}(\tau) &\equiv \chi_{\frac{i}{2}(nb+m/b)}^{(R)}(\tau) - \chi_{\frac{i}{2}(nb-m/b)}^{(R)}(\tau)
\end{aligned} \tag{2.26}$$

for degenerate representations. See appendix A for our definition of basic modular functions (like $\chi_0^{(NS)}(\tau)$) and its modular transformation.

The remaining wave functions ψ were then obtained from the conformal bootstrap method [12]. Here we show that they can also be obtained from the modular bootstrap method by considering the simplest degenerate primary operator $(1,2)_+$ and its fusion rule $\mathcal{N}_{i,j}^k$ with operators in R_+ sector. Actually, the fusion rule between $(1,2)_+$ and non-degenerate primary σ_{R+} is controllable [6, 7] and leads to

$$\begin{aligned}
n_{(1,2)_+, (n,m)_+}^h &= 2\mathcal{N}_{(1,2)_+, (n,m)_+}^h = 2\delta_{(n,m+1)_+}^h + 2\delta_{(n,m-1)_-}^h, \\
n_{(1,2)_+, \sigma_{R+}}^h &= 2\mathcal{N}_{(1,2)_+, \sigma_{R+}}^h = 2\delta_{(\sigma+ib/2)_{R+}}^h + 2\delta_{(\sigma-ib/2)_{R-}}^h.
\end{aligned} \tag{2.27}$$

Note that the chirality of the second terms is flipped due to the boundary Liouville action (see e.g., [12, 13]) whose fermion number is odd, and that the factor “2” follows from the wave functions of NSNS sector. The corresponding Cardy equations are

$$\begin{aligned}
&\langle (1,2)_+ | \nu_{RR}; \eta = +1 \rangle \langle \langle -\nu_{RR}; \eta = +1 | (1,2)_+ \rangle \\
&= \frac{1}{\sqrt{2}} \left(+S_{(1,3)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} - S_{(1,1)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} \right) \\
&= \frac{1}{\sqrt{2}} \frac{(4i \cosh(\pi b\nu) \sinh(2\pi\nu/b)) \cdot (4i \cosh(-\pi b\nu) \sinh(-2\pi\nu/b))}{4 \cosh(\pi b\nu) \cosh(\pi\nu/b)} \\
&\langle (n,m)_+ | \nu_{RR}; \eta = +1 \rangle \langle \langle -\nu_{RR}; \eta = +1 | (1,2)_+ \rangle \\
&= \frac{1}{\sqrt{2}} \left(+S_{(n,m+1)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} - S_{(n,m-1)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} \right) \\
&= \frac{1}{\sqrt{2}} \frac{(4i \sinh(n\pi b\nu + \frac{\pi n i}{2}) \sinh(m\pi\nu/b - \frac{\pi n i}{2})) \cdot (4i \cosh(-\pi b\nu) \sinh(-2\pi\nu/b))}{4 \cosh(\pi b\nu) \cosh(\pi\nu/b)} \\
&\langle \sigma_{R+} | \nu_{RR}; \eta = +1 \rangle \langle \langle -\nu_{RR}; \eta = +1 | (1,2)_+ \rangle \\
&= \frac{1}{\sqrt{2}} \left(+S_{(\sigma+ib/2)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} - S_{(\sigma-ib/2)_{\widetilde{NS}}}^{\nu_{\widetilde{R}}} \right) \\
&= \frac{1}{\sqrt{2}} \frac{(2i \sinh(2\pi i \sigma\nu)) \cdot (4i \cosh(-\pi b\nu) \sinh(-2\pi\nu/b))}{4 \cosh(\pi b\nu) \cosh(\pi\nu/b)},
\end{aligned} \tag{2.28}$$

and solving these equation we can obtain

$$\begin{aligned}\psi_{\sigma_{\bar{R}}}^{\nu_{\bar{R}}} &= 2i \sinh(2\pi i \sigma \nu), \\ \psi_{(n,m)_{\bar{R}}}^{\nu_{\bar{R}}} &= 4i \sinh\left(\pi n b \nu + \frac{i\pi n}{2}\right) \cosh\left(\frac{\pi m \nu}{b} - \frac{i\pi n}{2}\right),\end{aligned}\quad (2.29)$$

and this correctly reproduces the results of [12].

2.2.2 the (p, q) minimal superconformal field theory case

Next we apply the above argument to the case of (p, q) minimal superconformal field theory. This theory can be classified into two categories: even and odd models [54]. The modular matrices among NS, $\widetilde{\text{NS}}$ and R sector were derived in [48] from the character formula [55],

$$\begin{aligned}\chi_{(r,s)}^{(NS)}(\tau) &= \chi_0^{(NS)}(\tau) \sum_{n \in \mathbb{Z}} \left[q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}} \right], \\ \chi_{(r,s)}^{(\widetilde{NS})}(\tau) &= \chi_0^{(\widetilde{NS})}(\tau) \sum_{n \in \mathbb{Z}} (-1)^{2pq} \left[q^{\frac{(2npq+qr-ps)^2}{8pq}} - (-1)^{rs} q^{\frac{(2npq+qr+ps)^2}{8pq}} \right], \\ \chi_{(r,s)}^{(R)}(\tau) &= \chi_0^{(R)}(\tau) \sum_{n \in \mathbb{Z}} \left[q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}} \right], \\ \chi_{(\frac{p}{2}, \frac{q}{2})}^{(R)}(\tau) &= \frac{1}{2} \chi_0^{(R)}(\tau) \sum_{n \in \mathbb{Z}} \left[q^{\frac{(2npq)^2}{8pq}} - q^{\frac{(2npq+pq)^2}{8pq}} \right] \quad (\text{only even model}),\end{aligned}\quad (2.30)$$

as

$$S_{(r,s)}^{(\bar{r}, \bar{s})} = \frac{4}{\sqrt{pq}} (-1)^{\frac{(r-s)(\bar{r}-\bar{s})}{2}} \sin\left(\frac{r\bar{r}}{2p}(q-p)\pi\right) \sin\left(\frac{s\bar{s}}{2q}(q-p)\pi\right). \quad (2.31)$$

For the $\widetilde{\text{R}}$ wave function $\psi_{(r,s)}^{(\bar{r}, \bar{s})}$, we should know the fusion rule of open strings. One way to know is the super Coulomb gas formalism in Ramond sector [43]. Since the chirality of the screening charges is odd, The fusion rule is also given in the following form:

$$n_{(1,2)_+, (k,l)_+}^{(r,s)} = 2\mathcal{N}_{(1,2)_+, (k,l)_+}^{(r,s)} = 2\delta_{(k,l+1)_+}^{(r,s)} + 2\delta_{(k,l-1)_-}^{(r,s)}. \quad (2.32)$$

This leads to the Cardy equations,

$$\begin{aligned}\langle (1,2)_+ | (r,s)_{RR}; \eta = +1 \rangle \langle (-r, -s)_{RR}; \eta = +1 | (1,2)_+ \rangle \\ = \frac{1}{\sqrt{2}} \left(+S_{(1,3)_{\widetilde{\text{NS}}}}^{(r,s)_{\text{R}}} - S_{(1,1)_{\widetilde{\text{NS}}}}^{(r,s)_{\text{R}}} \right) \\ = \frac{-1}{\sqrt{2}} \frac{\left(\frac{4}{\sqrt{pq}} \sin\left(\frac{r}{2p}(q-p)\pi\right) \sin\left(\frac{2s}{2q}(q-p)\pi\right) \right)^2}{\frac{4}{\sqrt{pq}} \sin\left(\frac{r}{2p}(q-p)\pi\right) \sin\left(\frac{s}{2q}(q-p)\pi\right)},\end{aligned}$$

$$\begin{aligned}
& \langle (k, l)_+ | (r, s)_{RR}; \eta = +1 \rangle \langle (-r, -s)_{RR}; \eta = +1 | (1, 2)_+ \rangle \\
&= \frac{1}{\sqrt{2}} \left(S_{(k, l+1)_{\widetilde{NS}}}^{(r, s)_R} - S_{(k, l-1)_{\widetilde{NS}}}^{(r, s)_R} \right) \\
&= \frac{1}{\sqrt{2}} (-1)^{\frac{(k-l-1)(r-s)}{2}} \times \\
&\quad \times \frac{\left(\frac{4}{\sqrt{pq}} \sin\left(\frac{kr}{2p}(q-p)\pi\right) \sin\left(\frac{ls}{2q}(q-p)\pi\right) \right) \cdot \left(\frac{4}{\sqrt{pq}} \sin\left(\frac{r}{2p}(q-p)\pi\right) \sin\left(\frac{2s}{2q}(q-p)\pi\right) \right)}{\frac{4}{\sqrt{pq}} \sin\left(\frac{r}{2p}(q-p)\pi\right) \sin\left(\frac{s}{2q}(q-p)\pi\right)}.
\end{aligned} \tag{2.33}$$

Solving these equations, we obtain

$$\psi_{(k, l)}^{(r, s)} = \frac{4}{\sqrt{pq}} (-1)^{\frac{(k-l)(r-s)+1}{2}} \sin\left(\frac{kr}{2p}(q-p)\pi\right) \sin\left(\frac{ls}{2q}(q-p)\pi\right). \tag{2.34}$$

From this formula, we can see that any $(k, l) \neq (p/2, q/2)$ Cardy states have no contribution from the closed string $(r, s) = (p/2, q/2)$ state. From this consideration, we also conclude

$$\psi_{\theta_R}^{i_R} = \delta_{\theta_R}^{i_R} = S_{\theta_{\bar{R}}}^{i_{\bar{R}}}. \tag{2.35}$$

For an exercise of the later discussion, we also consider the spin-model GSO projections [41] for even models. Here we first assume $\Gamma|\theta_{RR}; \eta\rangle\rangle = -|\theta_{RR}; \eta\rangle\rangle$. We then do not have to reconsider the NS Cardy states $|h_{NS\pm}\rangle$ and we obtain

$$\begin{aligned}
|h_{NS\pm}\rangle &= \frac{1}{\sqrt{2}} \sum_{i \in NS} \frac{S_{h_{NS}}^{i_{NS}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = -1\rangle\rangle \pm \frac{1}{2^{3/4}} \sum_{i \in R} \frac{S_{h_{\widetilde{NS}}}^{i_R}}{\sqrt{S_{0_{\widetilde{NS}}}^{i_R}}} |i_{RR}; \eta = -1\rangle\rangle, \\
|h_{R+}\rangle &= \sum_{i \in NS} \frac{S_{h_R}^{i_{\widetilde{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle\rangle, \\
|\theta_{R\pm}\rangle &= \frac{1}{2} \sum_{i \in NS} \frac{S_{\theta_R}^{i_{\widetilde{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle\rangle \pm \frac{1}{2^{1/4}} \frac{1}{\sqrt{S_{0_{\widetilde{NS}}}^{i_R}}} |\theta_{RR}; \eta = +1\rangle\rangle.
\end{aligned} \tag{2.36}$$

For the case of $\Gamma|\theta_{RR}; \eta\rangle\rangle = +|\theta_{RR}; \eta\rangle\rangle$, the NS Cardy states $|h_{NS\pm}\rangle$ and $|\theta_{R\pm}\rangle$ are no longer Cardy states under the GSO projection. We should instead consider the following Cardy states:

$$\begin{aligned}
|h_{NS}\rangle &\equiv \frac{1}{\sqrt{2}} (|h_{NS+}\rangle + |h_{NS-}\rangle) = \sum_{i \in NS} \frac{S_{h_{NS}}^{i_{NS}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = -1\rangle\rangle, \\
|h_R\rangle &\equiv \frac{1}{\sqrt{2}} |h_{R\pm}\rangle = \frac{1}{\sqrt{2}} \sum_{i \in NS} \frac{S_{h_R}^{i_{\widetilde{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle\rangle, \\
|\theta_R\rangle &\equiv \frac{1}{\sqrt{2}} (|\theta_{R+}\rangle + |\theta_{R-}\rangle) = \frac{1}{\sqrt{2}} \sum_{i \in NS} \frac{S_{\theta_R}^{i_{\widetilde{NS}}}}{\sqrt{S_{0_{NS}}^{i_{NS}}}} |i_{NSNS}; \eta = +1\rangle\rangle.
\end{aligned} \tag{2.37}$$

Note that the corresponding fusion rule in this case is that of generalized Verlinde formula [52] (the fusion rule of super-Virasoro algebra) and the states propagating in the open channel are among super-Virasoro Verma module (not only among the Virasoro sub-Verma module). From this point of view, the factor $1/\sqrt{2}$ in the definition of the boundary states is necessary. It will be seen that the former case is like the type 0B GSO projection and the later case is like the type 0A GSO projection in minimal superstring theory. Although we can also consider other possibility (e.g., boundary states with $\eta \rightarrow -\eta$), we will not investigate this direction here.

2.3 Boundary states in (p, q) minimal superstring theory

We now combine the above Cardy states. Since the boundary states in (p, q) minimal superstring theory have been discussed in [29, 15, 32], we first of all summarize the important things given in the previous discussions.

1. Since the ghost Ishibashi states in RR sector are in $(-1/2, -3/2)$ picture⁹ and $(-1)^f = (-1)^{f_L+f_R} = (-1)^{f^L+f^M+1}$, we use the following type 0 GSO projection: $(-1)^{f^L+f^M} = +1$ (-1) for type 0B (type 0A) [29, 15].
2. Since $(-1)^{f^M} |\theta_{RR}; \eta\rangle\rangle = + |\theta_{RR}; \eta\rangle\rangle$, the closed-string contributions of Ramond ground states are denied in the boundary states of $\eta = +1$ ($\eta = -1$) branes in the type 0B (type 0A) cases [29, 15].
3. When we consider the case of negative $\mu < 0$, we should use the transformation $\mu \rightarrow -\mu$ and $\eta^{(L)} \rightarrow -\eta^{(L)}$ [15, 32]. That is, the boundary states of $\mu < 0$ are obtained with the following replacement in the wave functions of RR emissions:

$$\begin{aligned} S_{\sigma_{\widetilde{\text{NS}}}^{(L)}}^{\nu_{\text{R}}} &= (2 \cosh(2\pi i \nu \sigma)) \rightarrow (2 \cosh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2})), \\ \psi_{\sigma_{\widetilde{\text{R}}}^{(L)}}^{\nu_{\widetilde{\text{R}}}} &= (2i \sinh(2\pi i \nu \sigma)) \rightarrow (2i \sinh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2})), \end{aligned} \quad (2.38)$$

where $\epsilon = (1 - \text{sgn}(\mu))/2$. The corresponding boundary cosmological constants are chosen as follows:

$$\zeta = \begin{cases} \sqrt{|\mu|} \cosh(\pi b \sigma) & (\hat{\eta} = -1) \\ \sqrt{|\mu|} \sinh(\pi b \sigma) & (\hat{\eta} = +1) \end{cases} \quad (2.39)$$

with the parameter $\hat{\eta} = \eta \text{sgn}(\mu)$ [15].

⁹For a summary of ghost Ishibashi/Cardy states, see appendix B.

Considering this, the boundary states for type 0B theory are given as:¹⁰

Type 0B

$$\begin{aligned}
|\sigma, (k, l)_{\text{NS}\pm}; \eta = -1\rangle &= \\
&= \frac{1}{\sqrt{2}} \int_0^\infty d\nu \sum_{(r,s) \in \text{NSNS}} (2 \cosh(2\pi i \nu \sigma)) S_{(k,l)}^{(M)(r,s)} A_{\text{NS}}(\nu, (r, s)) |\nu, (r, s)_{\text{NSNS}}; \eta = -1\rangle \pm \\
&\quad \pm \frac{1}{2} \int_0^\infty d\nu \sum_{(r,s) \in \text{RR}} (2 \cosh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2})) S_{(k,l)}^{(M)(r,s)} A_{\text{R}}(\nu, (r, s)) |\nu, (r, s)_{\text{RR}}; \eta = -1\rangle, \\
&\quad (k + l \in 2\mathbb{Z}),
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
|\sigma, (k, l)_{\text{R}\pm}; \eta = +1\rangle &= \\
&= \frac{1}{\sqrt{2}} \int_0^\infty d\nu \sum_{(r,s) \in \text{NSNS}} (2 \cosh(2\pi i \nu \sigma)) S_{(k,l)}^{(M)(r,s)} A_{\text{NS}}(\nu, (r, s)) |\nu, (r, s)_{\text{NSNS}}; \eta = +1\rangle \pm \\
&\quad \pm \frac{1}{2} \int_0^\infty d\nu \sum_{(r,s) \in \text{RR}} (2i \sinh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2})) \psi_{(k,l)}^{(M)(r,s)} A_{\text{R}}(\nu, (r, s)) |\nu, (r, s)_{\text{RR}}; \eta = +1\rangle, \\
&\quad (k + l \in 2\mathbb{Z} + 1),
\end{aligned} \tag{2.41}$$

Here $(r, s) \in \text{NSNS}, \text{RR}$ means that (r, s) runs among

$$\begin{aligned}
\text{NSNS} : \quad & 1 \leq r \leq p-1, \quad 1 \leq s \leq q-1, \quad qr - ps > 0, \quad r + s \in 2\mathbb{Z}, \\
\text{RR} : \quad & 1 \leq r \leq p-1, \quad 1 \leq s \leq q-1, \quad qr - ps \geq 0, \quad r + s \in 2\mathbb{Z} + 1.
\end{aligned} \tag{2.42}$$

We define $A_{\text{X}}(\nu, (r, s)) \equiv A_{\text{X}}^{(L)}(\nu) \cdot A^{(M)}(r, s)$ ($\text{X} = \text{NS}$ or R) and

$$A^{(M)}(r, s) \equiv 1/\sqrt{S_{(1,1)}^{(M)(r,s)}}, \quad A^{(M)}(r, s) \cdot A^{(M)}(-r, -s) = 1/S_{(1,1)}^{(M)(r,s)}. \tag{2.43}$$

The Ishibashi states are

$$|\nu, (r, s)_{\text{XX}}; \eta\rangle = |\nu_{\text{XX}}; \eta\rangle \otimes |(r, s)_{\text{XX}}; \eta\rangle \otimes |Gh_{\text{XX}}; \eta\rangle. \tag{2.44}$$

The normalization of $\beta\gamma$ ghost Ishibashi/Cardy states is summarized in appendix B. The boundary states of ZZ branes are obtained by replacing the Liouville wave functions, for example,

$$S_{\sigma_{\text{NS}}}^{(L)} \nu_{\text{NS}} = (2 \cosh(2\pi i \nu \sigma)) \rightarrow S_{(n,m)_{\text{NS}}}^{(L)} \nu_{\text{NS}} = (4 \sinh(\pi n b \nu) \sinh(\pi m \nu / b)). \tag{2.45}$$

The normalization of $\eta = +1$ boundary states is the same as that of $\eta = -1$ boundary states.¹¹ It is due to the fact that open-channel spin fields $\Theta_{\pm}^{(L+M)}(z)$ form a doublet under

¹⁰Note that the Ramond Cardy state is given as $|\sigma, (p/2, q/2)_{\text{R}}\rangle = |\sigma, (p/2, q/2)_{\text{R}\pm}\rangle$.

¹¹ This should be compared with (2.16), where the normalization of $\eta = +1$ branes is $\sqrt{2}$ times bigger than that of $\eta = -1$ branes.

the combination of each Liouville and matter spin-field doublet, $\Theta_{\pm}^{(L)}(z)$ and $\Theta_{\pm}^{(M)}(z)$.¹² The case of matter Ramond ground states is also the same (the doublet is supplied from Liouville spin fields).

From this expression, we can identify the corresponding principal FZZT branes [15]. For $\eta = -1$ FZZT branes, we can show the relations modulo BRST,

$$|\sigma; (k, l)_{\text{NS}\pm}; \eta = -1\rangle = \sum_{m,n} \left| \sigma + \frac{i}{2}(mb - nb^{-1}); \eta = -1, \pm(-1)^{\frac{(k-l)-(m-n)}{2}} \right\rangle, \quad (2.46)$$

with $|\sigma; \eta = -1, \pm\rangle \equiv |\sigma; (1, 1)_{\text{NS}\pm}; \eta = -1\rangle$ following the arguments given in [15, 16]. For $\eta = +1$ FZZT branes, on the other hand, there is not such a principal brane among the above $\eta = +1$ FZZT branes. Instead, we actually show that the following $\eta = +1$ FZZT brane,

$$|\sigma; \eta = +1, \pm\rangle \equiv (-1)^{f_R} |\sigma; \eta = -1, \pm\rangle, \quad (2.47)$$

plays a role of a principal $\eta = +1$ FZZT brane:¹³

$$|\sigma; (k, l)_{\text{R}\pm}; \eta = +1\rangle = \sum_{m,n} \left| \sigma + \frac{i}{2}(mb - nb^{-1}); \eta = +1, \pm(-1)^{\frac{(k-l)-(m-n)}{2}} \right\rangle. \quad (2.48)$$

Because this principal $\eta = +1$ FZZT brane is not a Cardy state for the principal $\eta = -1$ FZZT branes, these principal branes cannot exist at the same time. Even though there is no open string spectrum that propagates among these two branes, they are necessary for the construction of all the spectrum of D-branes. Since these are simply related under the simple transformation $(-1)^{f_R}$, it is reminiscent of order/disorder parameters in the Kramers-Wanniers duality [56].¹⁴ Actually we argue in section 3 that the corresponding annulus amplitudes are not mutually local in spacetime.

The boundary states for type 0A theory are

Type 0A

$$\begin{aligned} |\sigma, (k, l)_{\text{NS}}; \eta = -1\rangle &= \\ &= \int_0^\infty d\nu \sum_{(r,s) \in \text{NSNS}} (2 \cosh(2\pi i \nu \sigma)) S_{(k,l)}^{(M)(r,s)} A_{\text{NS}}(\nu, (r, s)) |\nu, (r, s); \text{NSNS}; \eta = -1\rangle, \\ &\quad (k + l \in 2\mathbb{Z}), \end{aligned} \quad (2.49)$$

¹²From this fact, the relation $n_{ij}^k = \mathcal{N}_{ij}^k$ holds in the superstring case.

¹³Note that, in the Liouville wave functions of R-R sector, “sinh” turns to be “cosh” (“cosh” turns to be “sinh” if one considers $\mu < 0$) in this principal $\eta = +1$ FZZT branes.

¹⁴Note that this transformation is not the same as that of $\mu \rightarrow -\mu$ (or $\eta^{(L)} \rightarrow -\eta^{(L)}$).

$$\begin{aligned}
|\sigma, (k, l)_{\text{R}}; \eta = +1\rangle &= \\
&= \int_0^\infty d\nu \sum_{(r,s) \in \text{NSNS}} (2 \cosh(2\pi i \nu \sigma)) S_{(k,l)}^{(M)(r,s)} A_{\text{NS}}(\nu, (r, s)) |\nu, (r, s); \text{NSNS}; \eta = +1\rangle, \\
&\quad (k + l \in 2\mathbb{Z} + 1),
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
|\sigma; (\hat{p}, \hat{q})_{\text{R}\pm}; \eta = +1\rangle &= \\
&= \frac{1}{2} \int_0^\infty d\nu \sum_{(r,s) \in \text{NSNS}} (2 \cosh(2\pi i \nu \sigma)) S_{(k,l)}^{(M)(r,s)} A_{\text{NS}}(\nu, (r, s)) |\nu, (r, s); \text{NSNS}; \eta = +1\rangle \pm \\
&\quad \pm \frac{1}{\sqrt{2}} \int_0^\infty d\nu (2i \sinh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2})) A_{\text{R}}(\nu, (\hat{p}, \hat{q})) |\nu, (\hat{p}, \hat{q}); \text{RR}; \eta = +1\rangle.
\end{aligned} \tag{2.51}$$

The first two kinds of boundary states are given as¹⁵

$$|\sigma, (k, l)_{\text{X}}\rangle_{\mu>0}^{(0\text{A})} = \frac{1}{\sqrt{2}} \left(|\sigma, (k, l)_{\text{X}+}\rangle_{\mu>0}^{(0\text{B})} + |\sigma, (k, l)_{\text{X}-}\rangle_{\mu>0}^{(0\text{B})} \right). \tag{2.52}$$

The last boundary state of the Ramond ground state is defined by the following fusion rule:

$$n_{(\frac{p}{2}, \frac{q}{2})_+, (\frac{p}{2}, \frac{q}{2})_\pm}^{(r,s)} = \frac{1}{2} \mathcal{N}_{(\frac{p}{2}, \frac{q}{2})_+, (\frac{p}{2}, \frac{q}{2})_\pm}^{(r,s)} = \frac{1}{2} \sum_{n=1,2,\dots,p-1; m=1,3,\dots,q-1} \delta_{(n,m)_\pm}^{(r,s)}, \tag{2.53}$$

with the identification $(r, s) \sim (p - r, q - s)$. Under this normalization, we actually obtain the following nontrivial identification in the pure-supergravity case, $(p, q) = (2, 4)$:

$$\begin{aligned}
|\sigma, (1, 1)_{\text{NS}\pm}; \eta = -1\rangle_{\mu>0}^{(0\text{B})} &\simeq |\sigma, (1, 2)_{\text{R}\pm}; \eta = +1\rangle_{\mu<0}^{(0\text{A})}, \\
|\sigma, (1, 2)_{\text{R}}; \eta = +1\rangle_{\mu>0}^{(0\text{B})} &\simeq |\sigma, (1, 1)_{\text{NS}}; \eta = -1\rangle_{\mu<0}^{(0\text{A})},
\end{aligned} \tag{2.54}$$

argued in [29, 15, 32].

Although the following branes are the principal FZZT branes of type 0A theory:

$$|\sigma; \eta = -1\rangle \equiv |\sigma, (1, 1)_{\text{NS}}; \eta = -1\rangle, \quad |\sigma; \eta = +1\rangle \equiv (-1)^{f_{\text{R}}} |\sigma; \eta = -1\rangle, \tag{2.55}$$

the Cardy state of the Ramond ground state, $|\sigma; (\hat{p}, \hat{q}); \eta = +1\rangle$, cannot be written with the above principal FZZT branes. So we must treat these things separately.

3 Annulus amplitudes of the principal FZZT branes

In this section, we evaluate the annulus amplitudes of FZZT branes. Annulus amplitudes of ZZ branes are not considered here, since this kind of amplitude is obtained from those

¹⁵This normalization gives the natural oscillator algebra, $[\alpha_n^{[0]}, \alpha_m^{[0]}] = n\delta_{n+m}$, in the corresponding string field formulation [38].

of the principal FZZT branes (with the relation between ZZ and FZZT branes [14] in this case [15]).

Annulus amplitudes from the CFT approach has been studied in [14, 16] (bosonic cases) and [32, 17] (fermionic case). Our technical procedure follows them, and we will not repeatedly write such a thing. For the later convenience, we denote our amplitudes as

$$\begin{aligned}
Z_{\xi\xi'}^{\eta,\eta'}(\sigma, (k, l); \sigma', (k', l')) &\equiv \\
&\equiv \int_0^\infty dt \langle \sigma, (k, l); \eta = -1, \xi | q^{\frac{1}{2}(L_0 - \frac{c}{24})} \tilde{q}^{\frac{1}{2}(\bar{L}_0 - \frac{c}{24})} | \sigma', (k', l'); \eta = -1, \xi' \rangle \\
&\equiv Z_{\text{NSNS}}^{\eta,\eta'}(\sigma, (k, l); \sigma', (k', l')) + \xi\xi' Z_{\text{RR}}^{\eta,\eta'}(\sigma, (k, l); \sigma', (k', l')), \tag{3.1}
\end{aligned}$$

with $q = e^{-2\pi t}$. Throughout this analysis, we use the following relations that can be shown with the technique of [57]:

$$\begin{aligned}
&\frac{2}{\sqrt{pq}} \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2pq}} \sin\left(\frac{km}{p}\pi\right) \sin\left(\frac{lm}{q}\pi\right) \\
&= \begin{cases} \sum_{(r,s) \in \text{NSNS}} \sum_{n \in \mathbb{Z}} S_{(k,l)}^{(M)(r,s)} \left(q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}} \right), & (k-l \in 2\mathbb{Z}) \\ \sum_{(r,s) \in \text{NSNS}} \sum_{n \in \mathbb{Z}} S_{(k,l)}^{(M)(r,s)} (-1)^{npq} \left(q^{\frac{(2npq+qr-ps)^2}{8pq}} - (-1)^{rs} q^{\frac{(2npq+qr+ps)^2}{8pq}} \right), & (k-l \in 2\mathbb{Z} + 1), \end{cases} \\
&\frac{2}{\sqrt{pq}} \sum_{m \in \mathbb{Z} + pq/2} q^{\frac{m^2}{2pq}} \sin\left(\frac{km}{p}\pi + \frac{k}{2}\pi\right) \sin\left(\frac{lm}{q}\pi + \frac{l}{2}\pi\right) \\
&= \sum_{(r,s) \in \text{RR}} \sum_{n \in \mathbb{Z}} S_{(k,l)}^{(M)(r,s)} \left(q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}} \right), \quad (k-l \in 2\mathbb{Z}) \tag{3.2}
\end{aligned}$$

and the following useful formula that was used in [14]:

$$\sum_{m \in \mathbb{Z}} \frac{\cos(Am\pi)}{m^2 + \nu^2} = \frac{\pi \cosh((1-A)\nu\pi)}{\nu \sinh(\pi\nu)} \tag{3.3}$$

$$\sum_{m \in \mathbb{Z} + 1/2} \frac{\cos(Am\pi)}{m^2 + \nu^2} = \frac{\pi \sinh((1-A)\nu\pi)}{\nu \cosh(\pi\nu)}. \tag{3.4}$$

3.1 NSNS exchange amplitudes between FZZT branes

We now consider the general amplitudes of $Z_{\text{NSNS}}^{\eta,\eta'}(\sigma, (k, l); \sigma', (k', l'))$, so NSNS exchanges. This kind of amplitudes does not essentially depend on η and $\text{sgn}(\mu)$. The relation between

0A and 0B is the following:

$$\begin{aligned}
Z_{\text{NSNS}}^{(0A)\eta,\eta'}(\sigma, (k, l); \sigma', (k', l')) &= 2Z_{\text{NSNS}}^{(0B)\eta,\eta'}(\sigma, (k, l); \sigma', (k', l')), \quad ((k, l), (r, s) \neq (p/2, q/2)) \\
Z_{\text{NSNS}}^{(0A)\eta,+1}(\sigma, (k, l); \sigma', (p/2, q/2)) &= Z_{\text{NSNS}}^{(0B)\eta,+1}(\sigma, (k, l); \sigma', (p/2, q/2)), \quad ((k, l) \neq (p/2, q/2)) \\
Z_{\text{NSNS}}^{(0A)+1,+1}(\sigma, (p/2, q/2); \sigma', (p/2, q/2)) &= \frac{1}{2}Z_{\text{NSNS}}^{(0B)+1,+1}(\sigma, (p/2, q/2); \sigma', (p/2, q/2)). \quad (3.5)
\end{aligned}$$

So we only consider 0B theory, $Z_{\text{NSNS}}^{(0B)\eta,\eta'}(\sigma, (k, l); \sigma', (k', l'))$. Then 0B amplitudes we consider here are (i) NSNS amplitudes of $\eta\eta' = +1$ and (ii) NSNS amplitudes of $\eta\eta' = -1$. They can be expressed as follows:

$$\begin{aligned}
(i) \quad Z_{\text{NSNS}}^{(0B)\eta,\eta}(\sigma, (k, l); \sigma', (k', l')) \\
&= \frac{1}{2} \int_0^\infty d\nu \frac{\cosh(2\pi i \nu \sigma) \cosh(2\pi i \nu \sigma')}{\sinh(\pi \nu b) \sinh(\pi \nu / b)} \sum_{(r,s) \in \text{NSNS}} \frac{S_{(k,l)}^{(M)(r,s)} S_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} \times \\
&\quad \times \int_0^\infty dt \sum_{n \in \mathbb{Z}} q^{\frac{\sigma^2}{2}} (q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}}) \quad (3.6)
\end{aligned}$$

$$\begin{aligned}
(ii) \quad Z_{\text{NSNS}}^{(0B)\eta,-\eta}(\sigma, (k, l); \sigma', (k', l')) \\
&= -\frac{1}{2} \int_0^\infty d\nu \frac{\cosh(2\pi i \nu \sigma) \cosh(2\pi i \nu \sigma')}{\sinh(\pi \nu b) \sinh(\pi \nu / b)} \sum_{(r,s) \in \text{NSNS}} \frac{S_{(k,l)}^{(M)(r,s)} S_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} \times \\
&\quad \times \int_0^\infty dt \sum_{n \in \mathbb{Z}} (-1)^{npq} q^{\frac{\sigma^2}{2}} (q^{\frac{(2npq+qr-ps)^2}{8pq}} - (-1)^{rs} q^{\frac{(2npq+qr+ps)^2}{8pq}}) \quad (3.7)
\end{aligned}$$

Although they seem to be different forms, they come to be a unified form. We can actually reexpress the basic amplitudes $Z_{\text{NSNS}}^{\eta,\eta}(\sigma, (k, l); \sigma', (1, 1))$ by using (3.2) and (3.3) as follows:

$$\begin{aligned}
Z_{\text{NSNS}}^{\eta,-1}(\sigma, (k, l); \sigma', (1, 1)) \\
&= -\eta \int_0^\infty d\nu \frac{\cosh(2\pi i \nu \sigma) \cosh(2\pi i \nu \sigma')}{\sinh(\pi \nu b) \sinh(\pi \nu / b)} \times \frac{\sqrt{pq}}{\pi} \times \sum_{m \in \mathbb{Z}} \frac{\sin(\frac{km}{p}\pi) \sin(\frac{lm}{q}\pi)}{pq \nu^2 + m^2} \\
&= \frac{-\eta}{2} \int_{-\infty}^\infty \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu \sigma}{\sqrt{pq}}) \cosh(2\pi i \frac{\nu \sigma'}{\sqrt{pq}}) \sinh((\frac{p-k}{p})\pi \nu) \sinh(\frac{l}{q}\pi \nu)}{\sinh(\pi \nu / p) \sinh(\pi \nu / q) \sinh(\pi \nu)}. \quad (3.8)
\end{aligned}$$

By using the fusion relations for $r - s \in 2\mathbb{Z}$:

$$\frac{S_{(k,l)}^{(M)(r,s)} \cdot S_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} = \sum_{\substack{m=k+k'-1, k+k'-3, \dots, k-k'+1; \\ n=l+l'-1, l+l'-3, \dots, l-l'+1}} S_{(n,m)}^{(M)(r,s)}, \quad (3.9)$$

we obtain the following general formula

$$Z_{\text{NSNS}}^{\eta, \eta'}(\sigma, (k, l); \sigma', (k', l')) = \frac{\eta\eta'}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu\sigma}{\sqrt{pq}}) \cosh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}}) \sinh((\frac{p-k}{p})\pi\nu) \sinh(\frac{l}{q}\pi\nu) \sinh(\frac{k'}{p}\pi\nu) \sinh(\frac{l'}{q}\pi\nu)}{(\sinh(\pi\nu/p) \sinh(\pi\nu/q))^2 \sinh(\pi\nu)}. \quad (3.10)$$

Following the arguments of [16], we rewrite this amplitude as

$$= -\frac{\eta\eta'}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \cosh(2\pi i \frac{\nu\sigma}{\sqrt{pq}}) \cosh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}}) \frac{\sinh(\frac{k}{p}\pi\nu) \sinh(\frac{l}{q}\pi\nu) \sinh(\frac{k'}{p}\pi\nu) \sinh(\frac{l'}{q}\pi\nu)}{(\sinh(\pi\nu/p) \sinh(\pi\nu/q))^2} \times \left\{ \frac{\cosh(\pi\nu)}{\sinh(\pi\nu)} - \frac{\cosh(k\pi\nu/p)}{\sinh(k\pi\nu/p)} \right\} \quad (3.11)$$

The first term in the parenthesis is the main part of this amplitude, since the second term is actually a contribution from the unphysical poles of NSNS sector, $\nu = inp$ ($n \in \mathbb{Z}$),¹⁶ as is in the bosonic case [16]. The main part can be written with the amplitudes of the principal $\eta = \pm 1$ FZZT branes following the rule of (2.46) and (2.48):

$$Z_{\text{NSNS,main}}^{\eta, \eta'}(\sigma, (k, l); \sigma', (k', l')) = \sum_{m, n; m', n'} Z_{\text{NSNS,main}}^{\eta, \eta'}(\sigma + \frac{i}{2}(mb - n/b); \sigma' + \frac{i}{2}(m'b - n'/b)), \quad (3.12)$$

where the above basic amplitudes satisfy

$$Z_{\text{NSNS,main}}^{\eta, \eta}(\sigma; \sigma') = Z_{\text{NSNS,main}}^{-\eta, -\eta}(\sigma; \sigma') = -Z_{\text{NSNS,main}}^{\eta, -\eta}(\sigma; \sigma'). \quad (3.13)$$

The amplitudes of $\eta\eta' = +1$ principal FZZT branes can be evaluated with the procedure of [16] as

$$\begin{aligned} Z_{\text{NSNS}}^{\eta, \eta}(\sigma; \sigma') &= \frac{1}{2} \ln \left(\frac{\sinh(\pi \frac{\sigma + \sigma'}{\sqrt{pq}}) \sinh(\pi \frac{\sigma - \sigma'}{\sqrt{pq}})}{\sinh(p\pi \frac{\sigma + \sigma'}{\sqrt{pq}}) \sinh(p\pi \frac{\sigma - \sigma'}{\sqrt{pq}})} \right) \\ &= \frac{1}{2} \ln \left(\frac{\cosh(2\pi \frac{\sigma}{\sqrt{pq}}) - \cosh(2\pi \frac{\sigma'}{\sqrt{pq}})}{\cosh(2\pi \frac{p\sigma}{\sqrt{pq}}) - \cosh(2\pi \frac{p\sigma'}{\sqrt{pq}})} \right). \end{aligned} \quad (3.14)$$

¹⁶It can be easily seen by recalling the correspondence with differential operator $\mathbf{P}^{2n} = (\boldsymbol{\sigma} \mathbf{L}^{\hat{p}})^{2n}$ of 2-component KP hierarchy [38]. From the viewpoints of string field formulation [33, 34, 35, 38], this contribution comes from the normal ordering with respect to $\text{SL}(2, \mathbb{C})$ invariant vacuum of ζ [33].

At that time, we should be careful to treat the boundary cosmological constants ζ [12] and the uniformization parameter z [15]. We now define these parameters as

$$\tau \equiv \pi \frac{\sigma}{\sqrt{\hat{p}\hat{q}}}, \quad z = \cosh \tau, \quad \zeta = \begin{cases} \sqrt{|\mu|} \cosh(\hat{p}\tau) & (\hat{\eta} = -1) \\ \sqrt{|\mu|} \sinh(\hat{p}\tau) & (\hat{\eta} = +1) \end{cases}, \quad (3.15)$$

where we use the label (\hat{p}, \hat{q}) that is more suitable from the point of view of two-matrix models (or 2-component KP hierarchy) [38]:

$$\begin{aligned} (\hat{p}, \hat{q}) &= (p/2, q/2) \quad (\text{even model}), \\ &= (p, q) \quad (\text{odd model}). \end{aligned} \quad (3.16)$$

Therefore we obtain the following form:

$$Z_{\text{NSNS}}^{\eta, \eta}(\sigma; \sigma') = \begin{cases} \frac{1}{2} \ln \left(\frac{z - z'}{\zeta^2 - \zeta'^2} \right) & (\text{even model}) \\ \frac{1}{2} \ln \left(\frac{z^2 - z'^2}{\zeta^2 - \zeta'^2} \right) & (\text{odd model}). \end{cases} \quad (3.17)$$

Note that the dependence of ζ is changed as

$$z = \begin{cases} \frac{1}{2} \left((\zeta + \sqrt{\zeta^2 - \mu})^{1/\hat{p}} + (\zeta - \sqrt{\zeta^2 - \mu})^{1/\hat{p}} \right) \equiv z(\zeta) & (\hat{\eta} = -1), \\ \frac{1}{2} \left((\sqrt{\zeta^2 + \mu} + \zeta)^{1/\hat{p}} + (\sqrt{\zeta^2 + \mu} - \zeta)^{1/\hat{p}} \right) \equiv \tilde{z}(\zeta) & (\hat{\eta} = +1). \end{cases} \quad (3.18)$$

That is, ζ of $\eta = \pm 1$ FZZT branes are simply related as $\zeta_{\eta=-1}^2 - \zeta_{\eta=+1}^2 = \mu$.

The principal FZZT brane amplitudes of $\eta\eta' = -1$ are somewhat more complicated but we can say that the main parts are given as

$$Z_{\text{NSNS}, \text{main}}^{\eta, -\eta}(\sigma; \sigma') = \begin{cases} -\frac{1}{2} \ln(z - z') & (\text{even model}) \\ -\frac{1}{2} \ln(z^2 - z'^2) & (\text{odd model}). \end{cases}, \quad (3.19)$$

from the modulo-BRST equations (2.48). This means that if we have both brane operators $\psi_\eta(z)$ of $\eta = \pm 1$ principal branes, they should have the following behavior:¹⁷

$$\psi_{\eta=+1}(z') \psi_{\eta=-1}(z) \sim (z' - z)^{-1/2}, \quad (3.20)$$

since there are no RR exchange between these branes. In this sense, the principal $\eta = -1/+1$ branes are not mutually local in z (or ζ) spacetime coordinate.¹⁸ That is, their square-root

¹⁷Although we do not know the form of unphysical parts, these contributions are expected to be nothing but the normal ordering with respect to $\text{SL}(2, \mathbb{C})$ invariant vacuum of ζ [33].

¹⁸For the relations between a complex coordinate ζ and the 2-dimensional spacetime coordinate (ϕ_L, X_{FF}) of minimal string theory, see [35] and [30].

cut cannot dissolve in the asymptotic weak coupling region, $z \rightarrow \infty$ (or $\zeta \rightarrow \infty$). Note that this is due to the breaking of the Cardy consistency conditions. Actually the Cardy states of $\eta = +1$ FZZT branes do not have such a behavior, because they are linear combinations of the principal $\eta = +1$ FZZT branes, the number of which is even, and then the square root dissolves in the weak coupling region, $z \rightarrow \infty$ (or $\zeta \rightarrow \infty$). So we conclude that the principal $\eta = \pm 1$ FZZT branes in minimal superstring theory can be interpreted as order/disorder parameters in superstring spacetime.

3.2 RR exchange amplitudes between FZZT branes

Next we consider RR exchange amplitudes. Note that annulus amplitudes between $\eta = -1$ and $\eta = +1$ branes must vanish, $Z_{\text{RR}}^{\eta, -\eta} = 0$. It is because the superconformal residual symmetry is remained in cylinder and we need to insert a vertex operator to obtain non-zero results. Thus we now neglect this contribution. The amplitudes we consider are (iii) RR-ground state exchange amplitudes in 0A theory, (iv) RR amplitudes of $\eta = \mp 1$. First the case of (iii) is written as

$$\begin{aligned}
(iii) \quad Z_{\text{RR}}^{+,+}(\sigma, (\frac{p}{2}, \frac{q}{2}); \sigma', (\frac{p}{2}, \frac{q}{2})) \\
= -\frac{1}{2} \int_0^\infty d\nu \frac{\sinh(2\pi i \nu \sigma + \frac{\epsilon}{2} \pi i) \sinh(2\pi i \nu \sigma' + \frac{\epsilon}{2} \pi i)}{\cosh(\pi \nu b) \cosh(\pi \nu / b)} \times \\
\times \frac{\sqrt{pq}}{4} \int_0^\infty dt \sum_{n \in \mathbb{Z}} q^{\frac{\nu^2}{2}} (q^{\frac{(2npq)^2}{8pq}} - q^{\frac{(2npq+pq)^2}{8pq}}) \\
= -\frac{1}{2} \int_{-\infty}^\infty \frac{d\nu}{\nu} \frac{\sinh(2\pi i \frac{\nu \sigma}{\sqrt{pq}} + \frac{\epsilon}{2} \pi i) \sinh(2\pi i \frac{\nu \sigma'}{\sqrt{pq}} + \frac{\epsilon}{2} \pi i)}{4 \cosh(\pi \nu / p) \cosh(\pi \nu / q)} \frac{1}{\sinh(2\pi \nu / pq)}, \quad (3.21)
\end{aligned}$$

and the remaining case of (iv) is

$$\begin{aligned}
Z_{\text{RR}}^{-1,-1}(\sigma, (k, l); \sigma', (k', l')) \\
= -\frac{1}{2} \int_0^\infty d\nu \frac{\cosh(2\pi i \nu \sigma + \frac{\epsilon}{2} \pi i) \cosh(2\pi i \nu \sigma' + \frac{\epsilon}{2} \pi i)}{\cosh(\pi \nu b) \cosh(\pi \nu / b)} \sum_{(r,s) \in \text{RR}} \frac{S_{(k,l)}^{(M)(-r,-s)} S_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} \times \\
\times \int_0^\infty dt \sum_{n \in \mathbb{Z}} q^{\frac{\nu^2}{2}} (q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}}), \quad (3.22)
\end{aligned}$$

$$\begin{aligned}
Z_{\text{RR}}^{+,+}(\sigma, (k, l); \sigma', (k', l')) \\
= -\frac{1}{2} \int_0^\infty d\nu \frac{(i \sinh(2\pi i \nu \sigma + \frac{\epsilon}{2} \pi i)) (i \sinh(2\pi i \nu \sigma' + \frac{\epsilon}{2} \pi i))}{\cosh(\pi \nu b) \cosh(\pi \nu / b)} \sum_{(r,s) \in \text{RR}} \frac{\psi_{(k,l)}^{(M)(-r,-s)} \psi_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} \times
\end{aligned}$$

$$\times \int_0^\infty dt \sum_{n \in \mathbb{Z}} q^{\frac{\nu^2}{2}} (q^{\frac{(2npq+qr-ps)^2}{8pq}} - q^{\frac{(2npq+qr+ps)^2}{8pq}}), \quad (3.23)$$

The basic amplitudes $Z_{\text{RR}}^{-1,-1}(\sigma, (k, l); \sigma', (1, 1))$ ($k + l \in 2\mathbb{Z}$) are

$$\begin{aligned} & Z_{\text{RR}}^{-1,-1}(\sigma, (k, l); \sigma', (1, 1)) \\ &= - \int_0^\infty d\nu \frac{\cosh(2\pi i \nu \sigma + \frac{\epsilon \pi i}{2}) \cosh(2\pi i \nu \sigma' + \frac{\epsilon \pi i}{2})}{\cosh(\pi \nu b) \cosh(\pi \nu / b)} \times \\ & \quad \times \frac{\sqrt{pq}}{\pi} \sum_{m \in \mathbb{Z} + pq/2} \frac{\sin(\frac{km}{p}\pi + \frac{k\pi}{2}) \sin(\frac{lm}{q}\pi + \frac{l\pi}{2})}{pq \nu^2 + m^2} \\ &= \begin{cases} -\frac{1}{2} \int_{-\infty}^\infty \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu \sigma}{\sqrt{pq}} + \frac{\epsilon \pi i}{2}) \cosh(2\pi i \frac{\nu \sigma'}{\sqrt{pq}} + \frac{\epsilon \pi i}{2})}{\cosh(\pi \nu / p) \cosh(\pi \nu / q)} \times \\ \quad \times \frac{(-1)^{\frac{l-k}{2}} \sinh((\frac{p-k}{p})\pi \nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi \nu - \frac{l}{2}\pi i)}{\sinh(\pi \nu)} & \text{(even model)} \\ -\frac{1}{2} \int_{-\infty}^\infty \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu \sigma}{\sqrt{pq}} + \frac{\epsilon \pi i}{2}) \cosh(2\pi i \frac{\nu \sigma'}{\sqrt{pq}} + \frac{\epsilon \pi i}{2})}{\cosh(\pi \nu / p) \cosh(\pi \nu / q)} \times \\ \quad \times \frac{(-1)^{\frac{l-k}{2}} \cosh((\frac{p-k}{p})\pi \nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi \nu - \frac{l}{2}\pi i)}{\cosh(\pi \nu)} & \text{(odd model)} \end{cases} \quad (3.24) \end{aligned}$$

By using the fusion relations for $r - s \in 2\mathbb{Z} + 1$,¹⁹

$$\begin{aligned} \frac{S_{(k,l)}^{(M)(-r,-s)} \cdot S_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} &= \sum_{\substack{m=k+k'-1, k+k'-3, \dots, k-k'+1; \\ n=l+l'-1, l+l'-3, \dots, l-l'+1}} (-1)^{\frac{n-m}{2}} S_{(n,m)}^{(M)(r,s)}, \\ \frac{\psi_{(k,l)}^{(M)(-r,-s)} \cdot \psi_{(k',l')}^{(M)(r,s)}}{S_{(1,1)}^{(M)(r,s)}} &= \sum_{\substack{m=k+k'-1, k+k'-3, \dots, k-k'+1; \\ n=l+l'-1, l+l'-3, \dots, l-l'+1}} (-1)^{\frac{n-m}{2}} S_{(n,m)}^{(M)(r,s)}, \end{aligned} \quad (3.25)$$

we obtain the formula for the $\eta = -1$ case,

$$Z_{\text{RR}}^{-1,-1}(\sigma, (k, l); \sigma', (k', l'))$$

¹⁹Notice that $(-1)^{\frac{[(k-l)-(k'-l')](r-s)}{2}+1} = (-1)^{\frac{[(k+k')-(l+l')](r-s)}{2}}$.

$$\begin{aligned}
&= \left\{ \begin{aligned} &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu\sigma}{\sqrt{pq}} + \frac{\epsilon\pi i}{2}) \cosh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}} + \frac{\epsilon\pi i}{2})}{(\cosh(\pi\nu/p) \cosh(\pi\nu/q))^2} \times \\ &\quad \frac{\sinh((\frac{p-k}{p})\pi\nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi\nu - \frac{l}{2}\pi i) \sinh(\frac{k'}{p}\pi\nu - \frac{k'}{2}\pi i) \sinh(\frac{l'}{q}\pi\nu - \frac{l'}{2}\pi i)}{\sinh(\pi\nu)} \\ &\hspace{15em} \text{(even model)} \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{\cosh(2\pi i \frac{\nu\sigma}{\sqrt{pq}} + \frac{\epsilon\pi i}{2}) \cosh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}} + \frac{\epsilon\pi i}{2})}{(\cosh(\pi\nu/p) \cosh(\pi\nu/q))^2} \times \\ &\quad \frac{\cosh((\frac{p-k}{p})\pi\nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi\nu - \frac{l}{2}\pi i) \sinh(\frac{k'}{p}\pi\nu - \frac{k'}{2}\pi i) \sinh(\frac{l'}{q}\pi\nu - \frac{l'}{2}\pi i)}{\cosh(\pi\nu)} \\ &\hspace{15em} \text{(odd model)} \end{aligned} \right. \quad (3.26)
\end{aligned}$$

for the case of $\eta = +1$,

$$\begin{aligned}
&Z_{\text{RR}}^{+1,+1}(\sigma, (k, l); \sigma', (k', l')) \\
&= \left\{ \begin{aligned} &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{i \sinh(2\pi i \frac{\nu\sigma}{\sqrt{pq}} + \frac{\epsilon\pi i}{2}) \cdot i \sinh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}} + \frac{\epsilon\pi i}{2})}{(\cosh(\pi\nu/p) \cosh(\pi\nu/q))^2} \times \\ &\quad \frac{\sinh((\frac{p-k}{p})\pi\nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi\nu - \frac{l}{2}\pi i) \sinh(\frac{k'}{p}\pi\nu - \frac{k'}{2}\pi i) \sinh(\frac{l'}{q}\pi\nu - \frac{l'}{2}\pi i)}{\sinh(\pi\nu)} \\ &\hspace{15em} \text{(even model)} \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{i \sinh(2\pi i \frac{\nu\sigma}{\sqrt{pq}} + \frac{\epsilon\pi i}{2}) \cdot i \sinh(2\pi i \frac{\nu\sigma'}{\sqrt{pq}} + \frac{\epsilon\pi i}{2})}{(\cosh(\pi\nu/p) \cosh(\pi\nu/q))^2} \times \\ &\quad \frac{\cosh((\frac{p-k}{p})\pi\nu + \frac{k}{2}\pi i) \sinh(\frac{l}{q}\pi\nu - \frac{l}{2}\pi i) \sinh(\frac{k'}{p}\pi\nu - \frac{k'}{2}\pi i) \sinh(\frac{l'}{q}\pi\nu - \frac{l'}{2}\pi i)}{\cosh(\pi\nu)} \\ &\hspace{15em} \text{(odd model)} \end{aligned} \right. \quad (3.27)
\end{aligned}$$

Also in this case, we can separate the amplitudes into the sum of unphysical parts, $\nu = i(2n+1)\hat{p}$, and main parts. The main parts can be written with that of principal FZZT branes as

$$Z_{\text{RR},\text{main}}^{\eta,\eta}(\sigma, (k, l); \sigma', (k', l')) =$$

$$= \sum_{n,m;n',m'} (-1)^{\frac{[(k-l)-(n-m)]}{2} + \frac{[(k'-l')-(n'-m')]}{2}} Z_{\text{RR},\text{main}}^{\eta,\eta}(\sigma + \frac{i}{2}(nb - m/b); \sigma' + \frac{i}{2}(n'b - m'/b)), \quad (3.28)$$

and as is noted in section 2.2, (2.47), one can find that

$$Z_{\text{RR}}^{-1,-1}(\sigma; \sigma') = Z_{\text{RR}}^{+1,+1}(\sigma; \sigma'). \quad (3.29)$$

The amplitudes of principal FZZT branes are

(even model)

$$\begin{aligned} Z_{\text{RR}}^{-1,-1}(\sigma; \sigma')_{\mu>0} &= \frac{1}{2} \ln(\cosh(\tau) - \cosh(\tau')) + \frac{1}{2} \ln\left(\frac{\cosh(\hat{p}\tau) + \cosh(\hat{p}\tau')}{\cosh(\hat{p}\tau) - \cosh(\hat{p}\tau')}\right) \\ &= \frac{1}{2} \ln(z - z') + \frac{1}{2} \ln\left(\frac{\zeta + \zeta'}{\zeta - \zeta'}\right) \end{aligned} \quad (3.30)$$

$$\begin{aligned} Z_{\text{RR}}^{-1,-1}(\sigma; \sigma')_{\mu<0} &= \frac{1}{2} \ln(\cosh(\tau) - \cosh(\tau')) - \frac{1}{2} \ln\left(\frac{\sinh(\hat{p}\tau) + \sinh(\hat{p}\tau')}{\sinh(\hat{p}\tau) - \sinh(\hat{p}\tau')}\right) - \ln\left(\sinh\left(\frac{\tau + \tau'}{2}\right)\right) \\ &= \frac{1}{2} \ln(z - z') - \frac{1}{2} \ln\left(\frac{\zeta + \zeta'}{\zeta - \zeta'}\right) - \frac{1}{2} \ln\left(z z' + \sqrt{(z^2 - 1)(z'^2 - 1)} - 1\right) \end{aligned} \quad (3.31)$$

(odd model)

$$\begin{aligned} Z_{\text{RR}}^{-1,-1}(\sigma; \sigma')_{\mu>0} &= \frac{1}{2} \ln\left(\frac{\cosh(\tau) - \cosh(\tau')}{\cosh(\tau) + \cosh(\tau')}\right) + \frac{1}{2} \ln\left(\frac{\cosh(\hat{p}\tau) + \cosh(\hat{p}\tau')}{\cosh(\hat{p}\tau) - \cosh(\hat{p}\tau')}\right) \\ &= \frac{1}{2} \ln\left(\frac{z - z'}{z + z'}\right) + \frac{1}{2} \ln\left(\frac{\zeta + \zeta'}{\zeta - \zeta'}\right) \end{aligned} \quad (3.32)$$

$$\begin{aligned} Z_{\text{RR}}^{-1,-1}(\sigma; \sigma')_{\mu<0} &= \frac{1}{2} \ln\left(\frac{\sinh(\tau) - \sinh(\tau')}{\sinh(\tau) + \sinh(\tau')}\right) + \frac{1}{2} \ln\left(\frac{\sinh(\hat{p}\tau) + \sinh(\hat{p}\tau')}{\sinh(\hat{p}\tau) - \sinh(\hat{p}\tau')}\right) \\ &= \frac{1}{2} \ln\left(\frac{\sqrt{z^2 - 1} - \sqrt{z'^2 - 1}}{\sqrt{z^2 - 1} + \sqrt{z'^2 - 1}}\right) + \frac{1}{2} \ln\left(\frac{\zeta + \zeta'}{\zeta - \zeta'}\right) \end{aligned} \quad (3.33)$$

We then summarize the full amplitudes (NSNS + RR) of the principal $\eta = -1$ FZZT branes. For 0B theory,

(even model)

$$Z_{\xi_1, \xi_2}^{-1, -1}(\zeta_1; \zeta_2)_{\mu > 0} = \frac{1 + \xi_1 \xi_2}{2} \ln \left(\frac{z_1 - z_2}{\zeta_1 - \zeta_2} \right) - \frac{1 - \xi_1 \xi_2}{2} \ln(\zeta_1 + \zeta_2) \quad (3.34)$$

$$\begin{aligned} Z_{\xi_1, \xi_2}^{-1, -1}(\zeta_1; \zeta_2)_{\mu < 0} &= \frac{1 + \xi_1 \xi_2}{2} \ln \left(\frac{z_1 - z_2}{\zeta_1 - \zeta_2} \right) - \frac{1 - \xi_1 \xi_2}{2} \ln(\zeta_1 + \zeta_2) - \\ &\quad - \frac{\xi_1 \xi_2}{2} \ln \left(z_1 z_2 - \sqrt{(z_1^2 - 1)(z_2^2 - 1)} - 1 \right) \end{aligned} \quad (3.35)$$

(odd model)

$$Z_{\xi_1, \xi_2}^{-1, -1}(\zeta_1; \zeta_2)_{\mu > 0} = \frac{1 + \xi_1 \xi_2}{2} \ln \left(\frac{z_1 - z_2}{\zeta_1 - \zeta_2} \right) + \frac{1 - \xi_1 \xi_2}{2} \ln \left(\frac{z_1 + z_2}{\zeta_1 + \zeta_2} \right) \quad (3.36)$$

$$\begin{aligned} Z_{\xi_1, \xi_2}^{-1, -1}(\zeta_1; \zeta_2)_{\mu < 0} &= \\ &= \frac{1 + \xi_1 \xi_2}{2} \ln \left(\frac{\sqrt{z_1^2 - 1} - \sqrt{z_2^2 - 1}}{\zeta_1 - \zeta_2} \right) + \frac{1 - \xi_1 \xi_2}{2} \ln \left(\frac{\sqrt{z_1^2 - 1} + \sqrt{z_2^2 - 1}}{\zeta_1 + \zeta_2} \right), \end{aligned} \quad (3.37)$$

and for 0A theory,

$$Z_{\xi_1, \xi_2}^{-1, -1}(\zeta_1; \zeta_2) = \begin{cases} \ln \left(\frac{z - z'}{\zeta^2 - \zeta'^2} \right) & \text{(even model)} \\ \ln \left(\frac{z^2 - z'^2}{\zeta^2 - \zeta'^2} \right) & \text{(odd model)}, \end{cases}$$

are obtained.

Also for the pure-supergravity case of $(p, q) = (2, 4)$, (note that z and ζ accidentally coincide, $z = \zeta = \cosh \tau$), we can actually show that

$$\begin{aligned} Z_{\text{NSNS}}^{-1, -1}(\sigma_1, (1, 1); \sigma_2, (1, 1))_{\mu > 0} &= -\frac{1 - \xi_1 \xi_2}{2} \ln(\zeta_1 + \zeta_2), \\ Z_{\text{NSNS}}^{-1, -1}(\sigma_1, (1, 1); \sigma_2, (1, 1))_{\mu < 0} &= -\ln \left(\cosh \left(\frac{\tau_1 + \xi_1 \xi_2 \tau_2}{2} \right) \right) \\ &= -\frac{1}{2} \ln \left(\sqrt{(\zeta_1^2 - |\mu|)(\zeta_2^2 - |\mu|)} + \xi_1 \xi_2 \zeta_1 \zeta_2 - |\mu| \right). \end{aligned} \quad (3.38)$$

Of course, this is the previous results of this case [17].

4 Conclusion and discussion

In this paper, we investigate the explicit form of boundary states in (p, q) minimal superstring theory. For this purpose, we actually show the way to obtain all the wave functions of $\eta = \pm$ Cardy states within the modular bootstrap methods in $\mathcal{N} = 1$ superconformal field theory. We then identify the corresponding principal $\eta = \pm 1$ FZZT branes following

the arguments given in [15] and explicitly evaluate these annulus amplitudes. The principal $\eta = -1/+1$ FZZT branes of 0B theory are interpreted as order/disorder parameters which causes the Kramers-Wannier duality in the spacetime sense of this superstring theory.

From the analysis of [15], it was realized that among many different FZZT branes only a few numbers of principal FZZT branes are important and they correspond to the fundamental degrees of freedom of the theory. Since we can extract all the closed-string degrees of freedom from the principal $\eta = -1$ FZZT and its anti-FZZT branes [38], it is natural to think of the principal $\eta = -1$ FZZT brane as independent degrees of freedom.

In the case of Ising model, however, we can clearly see the relation with the fermion system by introducing the disorder parameter. In this sense, it is interesting if we could find some more general structures of minimal superstring theory, by considering how to describe the principal $\eta = +1$ FZZT branes in the exact nonperturbative formulations.

Of course this kind of duality is very familiar in conformal field theory, as the T-duality of worldsheet descriptions.²⁰ A new feature of our Kramers-Wannier like duality is that order/disorder parameters in minimal superstring theory correspond to D-branes in spacetime (not worldsheet observables). Since this structure is originated from the basic nature of the NSR formalism, it is interesting to investigate what is the spacetime properties of NSR superstring theory.

Acknowledgments

The author thanks M. Fukuma for useful discussions and encouragements, and K. Okuyama, K. Murakami, T. Takayanagi and M. Sato for useful conversations. The author also thanks H. Hata for various supports in preparing this paper. This work was supported in part by the Grant-in-Aid for the 21st Century COE “Center for Diversity and Universality in Physics” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. The author is also supported by Research Fellowships of the Japan Society for the Promotion of Science (JSPS) for Young Scientists (No. 18-2672).

²⁰See [58] for the investigations of the worldsheet Kramers-Wannier duality (T-duality) in non-critical string theory .

A Summary of the basic modular functions

Here we summarize our convention of the basic characters:

$$\begin{aligned}
\chi_0^{(NS)}(\tau) &\equiv \frac{q^{-1/48}}{\eta(\tau)} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = \frac{1}{\eta(\tau)} \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}, \\
\chi_0^{(\widetilde{NS})}(\tau) &\equiv \frac{q^{-1/48}}{\eta(\tau)} \prod_{n=1}^{\infty} (1 - q^{n-1/2}) = \frac{1}{\eta(\tau)} \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \\
\chi_0^{(R)}(\tau) &\equiv \frac{q^{1/24}}{\eta(\tau)} \prod_{n=0}^{\infty} (1 + q^n) = \frac{\sqrt{2}}{\eta(\tau)} \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}.
\end{aligned} \tag{A.1}$$

with Dedekind η -function $\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ and $q = e^{2\pi i \tau}$. $\theta_a(\tau)$ is the corresponding theta function. The modular transformations are

$$\begin{aligned}
\eta(-1/\tau) &= \sqrt{\frac{\tau}{i}} \eta(\tau), & \theta_3(-1/\tau) &= \sqrt{\frac{\tau}{i}} \theta_3(\tau), \\
\theta_4(-1/\tau) &= \sqrt{\frac{\tau}{i}} \theta_2(\tau), & \theta_2(-1/\tau) &= \sqrt{\frac{\tau}{i}} \theta_4(\tau).
\end{aligned} \tag{A.2}$$

B The Ishibashi/Cardy states of superconformal ghost

It is useful to note about our convention and notation of the Ishibashi/Cardy states of superconformal ghost [59]. Here we especially consider the normalization of $\beta\gamma$ Cardy states. It is convenient for $\beta\gamma$ ghost to be written in the form of [45]. We construct them with Watt's technique denoted in [50] as

$$|\beta\gamma_q; \eta\rangle\rangle = \sum_N |q; N\rangle \otimes U_\eta A \overline{|-2 - q; N^*\rangle} = U_\eta e^{-\sum_{r<0} (\gamma_r \tilde{\beta}_r + \tilde{\gamma}_r \beta_r)} |q\rangle \otimes \overline{|-2 - q\rangle} \tag{B.1}$$

where q is a corresponding picture and $|Gh_q; \eta\rangle\rangle$ is defined with the proper Ishibashi state of bc ghost, $|bc\rangle\rangle$, as $|Gh_q; \eta\rangle\rangle \equiv |\beta\gamma_q; \eta\rangle\rangle |bc\rangle\rangle$. We use the following hermitian conjugation,

$$\gamma_r^\dagger = -\gamma_{-r}, \quad \beta_r^\dagger = \beta_{-r}, \tag{B.2}$$

and we define the automorphism U_η to satisfy

$$U_\eta \tilde{\gamma}_r U_\eta^{-1} = -i\eta \tilde{\gamma}_r, \quad U_\eta \tilde{\beta}_r U_\eta^{-1} = i\eta \tilde{\beta}_r, \quad U_\eta^\dagger = -(-1)^{f_R} \cdot U_\eta^{-1}, \quad U_{-\eta} = U_\eta (-1)^{f_R}. \tag{B.3}$$

The normalization of states is defined from the open string character summed over the picture q Hilbert space $\mathcal{H}^{(q)}$ as follows:

$$\begin{aligned}
\langle\langle\beta\gamma_{-1};\eta|T(q)|\beta\gamma_{-1};\eta\rangle\rangle &= +\frac{q^{1/24}}{\prod_{n\geq 1}(1+q^{n-1/2})^2} \\
&= +\frac{\tilde{q}^{1/24}}{\prod_{n\geq 1}(1+\tilde{q}^{n-1/2})^2} = -\text{tr}_{\mathcal{H}_{\beta\gamma}^{(-1)}}[(-1)^{f_R}\tilde{q}^{L_0-c/24}] \\
\langle\langle\beta\gamma_{-3/2};\eta|T(q)|\beta\gamma_{-1/2};\eta\rangle\rangle &= -\frac{q^{-1/12}}{2\prod_{n\geq 1}(1+q^n)^2} \\
&= -\frac{\tilde{q}^{1/24}}{\prod_{n\geq 1}(1-\tilde{q}^{n-1/2})^2} = -\text{tr}_{\mathcal{H}_{\beta\gamma}^{(-1)}}[\tilde{q}^{L_0-c/24}] \\
\langle\langle\beta\gamma_{-1};\eta|T(q)(-1)^{f_R}|\beta\gamma_{-1};\eta\rangle\rangle &= -\frac{q^{1/24}}{\prod_{n\geq 1}(1-q^{n-1/2})^2} \\
&= -\frac{\tilde{q}^{-1/12}}{2\prod_{n\geq 1}(1+\tilde{q}^n)^2} = -i\text{tr}_{\mathcal{H}_{\beta\gamma}^{(-1/2)}}[(-1)^{f_R}\tilde{q}^{L_0-c/24}],
\end{aligned} \tag{B.4}$$

where $T(q) = q^{\frac{1}{2}(L_0-c/24)}\bar{q}^{\frac{1}{2}(\bar{L}_0-c/24)}$. The normalization of the second equation follows that of the first equation.²¹ Note that the normalization (or the sign) of the third equation is required from the spacetime statistics in superstring theory (open strings in R sector are fermions) and that this negative sign of the character can be consistently obtained from the definition $|Gh_q; -\eta\rangle\rangle = (-1)^{f_R}|Gh_q; \eta\rangle\rangle$ of the closed-channel Ishibashi states.

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²¹ If we assume that the character in NS sector is expanded as $\tilde{q}^{1/24} + \dots$, then the character in $\widetilde{\text{NS}}$ sector must be expanded as $-\tilde{q}^{1/24} + \dots$.

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